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ABSTRACT: This work represents an investigation of the normal solvability of Abel's generalized integral equation from the standpoint of its relationship to the Cauchy kernel equation, where all functions are assumed to be real. The relations involved in this study are presented and the pertinent equations are derived.

In this study we will investigate the normal solvability of the generalized integral Abel equation /1019[1]

$$M\varphi \equiv u(x)\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\mu} + v(x)\int_x^b \frac{\varphi(t)\,dt}{(t-x)^\mu} + T\varphi = F(x) \tag{1}$$

and of its conjugate equation

$$M^*\psi \equiv \int_a^x \frac{v(t)\psi(t)}{(x-t)^\mu}\,dt + \int_x^b \frac{u(t)\psi(t)}{(t-x)^\mu}\,dt + T^*\psi = F_1(x), \tag{2}$$

making use of the relationship of these equations to the Cauchy kernel equation. Equation (1) was first solved for T = 0 by K. D. Sakalyuk [5]. The rigid limitations imposed in reference [5] on u, v, and F were relaxed by L.B. Wolfersdorf [6]. F.V. Chumokev [7] also obtained several results for the complete equation (1), (under the rigid assumptions relative to T, u, v, and F). Henceforth all functions will be considered as real functions.

[1] Numbers in the margin indicate pagination in the foreign text.

I. Let $0 < \alpha < 1$ and $a \leqslant x \leqslant b$. We will then make the following definitions:

$$I_{ax}^{\alpha}\varphi \equiv \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\varphi(t)\,dt}{(x-t)^{1-\alpha}}, \quad I_{xb}^{\alpha}\varphi \equiv \frac{1}{\Gamma(\alpha)}\int_x^b \frac{\varphi(t)\,dt}{(t-x)^{1-\alpha}}; \tag{3}$$

$$A_{\alpha}\varphi \equiv \int_a^b \frac{\varphi(t)\,dt}{|x-t|^{1-\alpha}}, \quad B_{\alpha}\psi \equiv \int_a^b \frac{\operatorname{sign}(x-t)}{|x-t|^{1-\alpha}}\varphi(t)\,dt; \tag{4}$$

$$S\varphi \equiv \frac{1}{\pi}\int_a^b \frac{\varphi(t)}{t-x}\,dt; \tag{5}$$

$$r_a\varphi \equiv (x-a)\varphi(x), \qquad r_b\varphi \equiv (b-x)\varphi(x), \qquad r\varphi \equiv (x-a)(b-x)\varphi(x).$$

Lemma 1. Operators (3) - (5) are related by the identities

$$B_{\alpha}\varphi \equiv -\operatorname{tg}\left(\frac{\alpha\pi}{2}\right) A_{\alpha}\left(\frac{1}{r^{\alpha/2}} S r^{\alpha/2}\varphi\right) \equiv -\operatorname{tg}\left(\frac{\alpha\pi}{2}\right) r^{\alpha/2} S\left(\frac{1}{r^{\alpha/2}} A_{\alpha}\varphi\right); \tag{6}$$

$$A_{\alpha}\varphi \equiv \operatorname{ctg}\left(\frac{\alpha\pi}{2}\right) B_{\alpha}\left(\frac{1}{r^{(1+\alpha)/2}} S r^{(1+\alpha)/2}\varphi\right) \equiv \operatorname{ctg}\left(\frac{\alpha\pi}{2}\right) r^{(1+\alpha)/2} S\left(\frac{1}{r^{(1+\alpha)/2}} B_{\alpha}\varphi\right); \tag{7}$$

$$I_{xb}^{\alpha}\varphi \equiv \cos(\alpha\pi)\, I_{ax}^{\alpha}\varphi + \sin(\alpha\pi)\, I_{ax}^{\alpha}\frac{1}{r_a^{\alpha}} S r_a^{\alpha}\varphi; \tag{8}$$

$$I_{ax}^{\alpha}\varphi \equiv \cos(\alpha\pi)\, I_{xb}^{\alpha}\varphi - \sin(\alpha\pi)\, I_{xb}^{\alpha}\frac{1}{r_b^{\alpha}} S r_b^{\alpha}\varphi; \tag{9}$$

$$r_b^{\alpha} S \frac{1}{r_b^{\alpha}} I_{ax}^{\alpha}\varphi \equiv I_{ax}^{\alpha}\frac{1}{r_a^{\alpha}} S r_a^{\alpha}\varphi; \tag{10}$$

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The proof of identities (6) - (11) is based on the fact that after the interchange of the order of integration in the iterated integrals in (6) - (11), then intrinsic integrals are obtained which are easily expressed through elementary functions[1]. Identities (6) - (11) are valid[2] if $\phi \in L_p(\rho)$, where $L_p(\rho)$ is a class of functions that are summable in $[a, b]$ to the $p > 1$ power relative to the weight $\rho(t)$ and used in the theory of the Cauchy kernel equation [3].

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[1] For (8) - (11) we may use formula 3.228 from reference [8]. For (6) - (7) we obtain intrinsic integrals by solving the following equations:

$$\operatorname{sign}(\tau - x)\operatorname{ctg}(\alpha\pi/2)\varphi(x) - S\varphi = 0, \quad \operatorname{tg}(\alpha\pi/2)\psi(x) - S(\operatorname{sign}(t-\tau)\psi(t)) = 0.$$

[2] For all $x \in (a, b)$ if $p > 1/\alpha$ and almost everywhere if $p \leqslant 1/\alpha$.

II. By using the relationships (6) - (11), equations (1) and (2) are reduced to congugate equations with the Cauchy kernel. We will consider three means by which this reduction is accomplished.

A. By writing (1) - (2) in the form

$$uI_{ax}^{1-\mu}\varphi + vI_{xb}^{1-\mu}\varphi + T\varphi = \frac{1}{\Gamma(1-\mu)}F; \tag{1'}$$

$$I_{ax}^{1-\mu}(v\psi) + I_{xb}^{1-\mu}(u\psi) + T^{*}\psi = \frac{1}{\Gamma(1-\mu)}F_1 \tag{2'}$$

and applying identities (8) and (10) to (1') and (9) to (2') we obtain

$$a_1\Phi + a_2 r_b^{1-\mu} S \frac{1}{r_b^{1-\mu}}\Phi + K\Phi = F; \tag{12}$$

$$a_1\psi - \frac{1}{r_b^{1-\mu}} S r_b^{1-\mu} a_2\psi + K^{*}\psi = f_1, \tag{13}$$

where

$K = TI_{ax}^{-(1-\mu)}$, $\Phi = I_{ax}^{1-\mu}\varphi$, $f_1 = I_{ax}^{-(1-\mu)}F_1$[1] $a_1(x) = \Gamma(1-\mu)[u(x) - v(x)\cos\mu\pi]$, $a_2(x) = \Gamma(1-\mu)\sin(\mu\pi)v(x)$.

Operator T must satisfy the condition

$$TI_{ax}^{-(1-\mu)} = T_1, \tag{14}$$

where T_1 is a completely continuous operator in the given space of the functions of Φ (explained below).

B. If, however, we apply (9) and (11) to (1') and (8) to (2') we will obtain

$$b_1\chi - b_2 r_a^{1-\mu} S \frac{1}{r_a^{1-\mu}}\chi + K_1\chi = F,$$

$$b_1\psi + \frac{1}{r_a^{1-\mu}} S r_a^{1-\mu} b_2\psi + K_1^{*}\psi = f_2,$$

[1] $I_{ax}^{-(1-\mu)}$ is the operator of fractional differentiation, the inverse of $I_{ax}^{(1-\mu)}$.

where

$$K_1 = TI_{tb}^{-(1-\mu)},\ \chi = I_{xb}^{1-\mu}\varphi,\ f_2 = I_{xb}^{-(1-\mu)}F_1, \qquad b_1(x) = \Gamma(1-\mu)[v(x) - u(x)\cos\mu\pi],\ b_2(x) = \Gamma(1-\mu)\sin(\mu\pi)u(x),$$

and, as in (14), we should require that

$$TI_{tb}^{-(1-\mu)} = T_1. \tag{14'}$$

C. Now, by writing (1) - (2) in the form

$$\frac{u+v}{2}A_{1-\mu}\varphi + \frac{u-v}{2}B_{1-\mu}\varphi + T\varphi = F; \tag{1''}$$

$$A_{1-\mu}\left(\frac{u+v}{2}\psi\right) + B_{1-\mu}\left(\frac{u-v}{2}\psi\right) + T^*\psi = F_1 \tag{2''}$$

and applying (6) to (1") and (2"), we have /1021

$$d_1\Omega - d_2 r^{(1-\mu)/2} S \frac{1}{r^{(1-\mu)/2}}\Omega + K_2\Omega = F,$$

$$d_1\psi + \frac{1}{r^{(1-\mu)/2}} S r^{(1-\mu)/2}\psi + K_2^*\psi = f_3,$$

where

$$K_2 = TA_{1-\mu}^{-1},\quad \Omega = A_{1-\mu}\varphi,\quad f_3 = A_{1-\mu}^{-1}F_1,\quad d_1 = (u+v)/2,\quad d_2 = \operatorname{ctg}(\mu\pi/2)(u-v)/2.$$

We require here that

$$TA_{1-\mu}^{-1} = T_1. \tag{14''}$$

It can be shown by using identities (6) - (11) that the requirements (14), (14'), and (14") are equivalent. The simple sufficient condition for satisfying (14), (14') and (14") is defined below.

Lemma 2. Let

$$T = \int_a^b T(x, t)\varphi(t)\,dt,$$

$$T(x, t) = \begin{cases} c_1(x, t)(x - t)^{-\nu_1}, & t < x, \\ c_2(x, t)(t - x)^{-\nu_2}, & t > x, \end{cases}$$

where $$0 \leqslant \nu_i < \mu, \ |\partial c_i / \partial t| < \mathrm{const}/|x - t|, \ i = 1, 2.$$

Then the kernels of operators (14), (14'), and (14") are represented as the sum of the degenerate kernel and the kernel with weak singularity.

We will note that methods A and B differ only slightly. They are preferable to method C, since their solution requires only that we solve the Abel equation (classic), whereas with the third method we are required to solve the more complex equation, namely, $A_{1-\mu}\phi' = \Omega$.

III. Operator M is completely continuous and, therefore, has no limited regularization. Consequently [4], M is not a Noether operator in the usual sense. For equations (1) and (2), however, the Noether theorems may be encountered in special spaces.

The conclusions below are made on the basis of the reduction in II of equations (1) and (2) to equations with the Cauchy kernel. Let X and Y be spaces of functions in which the Noether theorem is applicable to equations (12) and (13). (For example, X and Y are classes of Holder functions combined into (a, b) [1, 2] or conjugate spaces $L_p(\rho)$ and $L_q(\rho^{1-q})$ [3].) Let X first be such that there exists $p > 1$ and weight $\rho(t)$ (exactly as in reference [3], page 12) for which $\mathcal{L}_p(\rho) \supseteq I_{ax}^{-(1-\mu)}(\bar{X})$.

We will then designate $B_X = I_{ax}^{-(1-\mu)}(X)$. The following is then valid.[1]

Theorem 1. Let

$$u(x),\ v(x) \in H^\lambda,\ \lambda > 1 - \mu, \ \text{и} \ F(x) \in X,\ f_1(x) \in Y.$$

The Noether theorems for equations (1) and (2) will be satisfied if the solutions of (1) are found in the space B_X and the solutions of (2) are found in Y. The subscript of equation (1) in this case is equal to that of equation (12).

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[1] Assuming that $u^2(x) + v^2(x) \neq 0$.

In Lemma 3 we will give the necessary criterion such that

$$I_{ax}^{-(1-\mu)}(X) \subseteq \mathcal{L}_p(\rho).$$

Lemma 3. Let $u(x),\ v(x),\ F(x) \in H^{\lambda},\ \lambda > 1-\mu,$ and let

$$\gamma(x) = \frac{1}{2\pi i} \ln \frac{G(x-0)}{G(x+0)},$$

where

$$G(x) = \begin{cases} (u - e^{\mu\pi i} v)(u - e^{-\mu\pi i} v)^{-1}, & x \in [a,b] \\ 1, & x \overline{\in} [a,b] \end{cases}.$$

If $-\mu < \gamma(a) < 1,\ -1 < \gamma(b) < \mu,$ then all Holder solutions /1022

of Φ in (a,b) of equation (12) are represented in the form

$$\Phi = I_{ax}^{1-\mu}\varphi,\quad \varphi(x) = (x-a)^{\nu_a}(b-x)^{\nu_b}\varphi_0(x),\quad \varphi_0(x) \in H^{\lambda+\mu-1},$$
$$\nu_a = \min(\mu - 1,\ \mu - 1 + \gamma(a)),\quad \nu_b = \min(0, \gamma(b)).$$

We may eliminate the requirement that u(x) and v(x) be smooth by assuming that the solution of (1) is in the class of generalized functions. Now

$X = \mathcal{L}_p(\rho),\ p > 1$ and $\rho(t) = (b-t)^{-p(1-\mu)}\rho_0(t),$ where the weight of $\rho_0(t)$ is taken from the function G(i), following the example of reference [3], page 85, and let B be the space of generalized fractional derivatives of the $1-\mu$ power from the functions of $L_p(\rho)$ (of the functionals $\phi = \Phi^{(1-\mu)}$ over the class of basic functions $\Psi(x)$ of the form $\Psi(x) = I_{xb}^{1-\mu}\psi,\ \psi \in L_q(\rho^{1-q})$):

$$(\varphi, \Psi) = (I_{ax}^{-1+\mu}\Phi, \Psi) = (\Phi, I_{xb}^{-1+\mu}\Psi) = (\Phi, \psi).$$

By determining the operators (3) - (5) in the space B in the required manner, it is easy to show that identities (6) - (11) are also valid for $\phi \in B$ and, consequently, the reduction to equation (12) remains valid.

Theorem 2 yields a basic result.

Theorem 2. Let u(x), v(x) be continuous to [a,b] $f_1 \in L_q(\rho^{1-q})$, $F(x) \in L_p(\rho)$ and let T be a completely continuous operator from B in $L_p(\rho)$. Then, for equation (1) and (2), the Noether theorems are satisfied in the spaces B and $L_q(\rho^{1-q})$ respectively.

Theorem 3. Let R_s be the regulator of equation (12). Then the operator $R = I_{ax}^{-1+\mu} R_s$ is the regulator (both left and right) of operator M, whereupon regularization from the left results in an equation which is regular in B: $RM\varphi \equiv \varphi + T_B\varphi = RF$, and regularization from the right results in an equation which is regular in $\mathcal{L}_p(\rho)$: $MR\Phi = \Phi + T_L\Phi = F$; T_B and T_L are completely continuous operators in B and $L_p(\rho)$, respectively.

In conclusion I wish to express my sincere gratitude to Professor F.D. Gakhov, who supervised this work.

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